ESSENTIAL MATHEMATICS

CONCISE EDITION

Worksheets 1–18: Foundations

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Preface

The present book aims to synthesize the main concepts of mathematics from the high school curriculum.

The motivation for creating this work was the need for a single book containing mathematical formulas, solution methods, and typical exercises, for which it would otherwise be necessary to consult all high school textbooks and many problem collections. The goal is to provide a succinct and focused presentation of the main concepts, without claiming to replace work from textbooks, exam problem books or specialized collections. Rather, it aims to be both an introduction to the study of high school mathematics and a reference manual for quick consultation of the main formulas and solution methods.

The topics covered are numbered according to the system Sheet 1, Sheet 2, Sheet 3, and so on.

In order to prepare for various exams, it is recommended that after completing the present material, preparation be done using a collection of exam subjects for the respective type of exam, with the aim of developing the specific skills for that type of examination.

As an example in this regard, exam subjects from previous years may be useful.

The Author

HOW TO USE THIS BOOK

Frequently Asked Questions

Q: Who is this book for?

A: This concise edition serves multiple audiences:

a) Students seeking foundational mastery

- Essential formulas, methods, and strategies for high school mathematics
- Perfect for building solid foundations before advanced study
- Complete coverage of standard international curriculum

b) Parents helping their children

- Quickly recall fundamental concepts and methods
- Clear explanations you can pass on to your child
- Organized systematically for easy reference

c) Students wanting rigorous understanding

- Not just "HOW" but "WHY" — develop mathematical thinking
- Build foundations that make advanced mathematics easier
- Self-study optimized — learn independently with confidence

d) Diaspora families

- Available in both **Romanian** and **English** editions
- Preserve mathematical rigor while living abroad
- Help children excel in any educational system

Q: What mathematical level does this book cover?

A: This book covers **essential high school mathematics** — the foundation every serious student needs.

Content (Worksheets 1-18):

- Geometry and trigonometry fundamentals
- Complex numbers and algebraic techniques
- Exponentials, logarithms, and combinatorics
- Polynomials and equations

Difficulty rating: xxx (3/5) — Intermediate

This material represents what a **rigorous European high school** teaches in grades 9-10. In many countries, this is the complete high school mathematics curriculum. In systems with more advanced tracks, this forms the essential foundation for further study.

Q: Can I print this book?

A: Yes! This book is formatted for standard **A4 paper**. You can print individual worksheets or the entire book on any home or office printer.

Many students find printed worksheets helpful for quick formula reference during homework or exam preparation. Keep them in a binder for easy access.

Q: What was the intention behind this book?

A: To provide **exceptional quality at zero cost** — like giving a **samurai sword** to those who need it most.

Quality mathematical education should not depend on family income or geographic location. This book offers **rigorous, proven methodology** developed over 45 years of teaching — freely available to everyone.

The principle:

- Not a "cheap" book because it's free
- But a **premium resource** made accessible to all
- Excellence in education is a right, not a privilege

Core philosophy: Hard work + quality resources = success. This book provides the quality resource. **The hard work is up to you.**

This book is your "samurai sword" — a tool of exceptional quality, given freely. What you achieve with it depends entirely on you.

*Note: For students seeking university-level preparation including calculus, linear algebra, and advanced topics, see the **Advanced Edition** (Worksheets 1–34).*

Good luck!

Table of Contents

SHEET 1: Plane Geometry

Terms Used:

The following terms are considered to be known from middle school. Make sure, however, that they are 100% clear to you:

- line, ray, segment, perpendicular bisector of a segment.
- parallel lines, secant line, alternate interior angles, alternate exterior angles, corresponding angles
- angle bisector
- symmetric: symmetric of a point with respect to a line, symmetric of a line with respect to a line.
- angles: acute, obtuse, complementary, supplementary
- triangle: arbitrary triangle, isosceles, equilateral, right triangle, congruent triangles (cases of congruence),
- triangle: similar triangles (cases of similarity), Thales' theorem.
- triangle: sum of angles in a triangle, exterior angle of a triangle (what it is and how to calculate it),
- triangle: midline (what it is and how to calculate it)

Important Lines in a Triangle:

• Angle Bisector: Divides an angle into two equal parts. Any point on the bisector is equidistant from the sides of the angle. The angle bisectors of a triangle intersect at the center of the inscribed circle.

• Median: Connects the vertex of a triangle with the midpoint of the opposite side. The medians of a triangle intersect at the center of gravity G of the triangle. G is located two-thirds from the vertex and one-third from the base.

• Perpendicular Bisector: Is perpendicular to the midpoint of a segment. Any point on the perpendicular bisector is equidistant from the endpoints of the segment. The perpendicular bisectors of a triangle intersect at the center of the circumscribed circle of the triangle.

• Altitude: Is the perpendicular drawn from a vertex of the triangle to the opposite side. The altitudes of a triangle intersect at the orthocenter of the triangle ("Ortho" means right in Greek)

Area of an Arbitrary Triangle:

(1) S= $\frac{base*altitude}{2}$

(2) S= $\frac{b*c*sin(A)}{2}$

(3) S= $\frac{a*b*c}{4*R}$, where R= radius of the circumscribed circle (intersection of perpendicular bisectors)

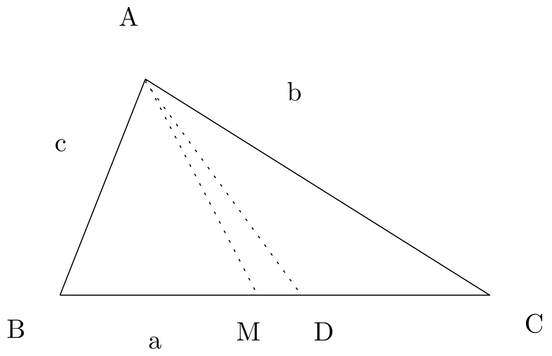
(4) $S = r * p$, where r= radius of the inscribed circle (intersection of angle bisectors), p= semiperimeter=$\frac{a+b+c}{2}$

(5) S= $\sqrt{p*(p-a)*(p-b)*(p-c)}$, where p=semiperimeter. (Heron's Formula)

(6) Useful idea: Expressing the area in two ways, in problems, and finding the unknown

Theorems in the Arbitrary Triangle

Let ABC be an arbitrary triangle, AD= median, AM= angle bisector



•Law of Sines: $\frac{a}{sin(A)} = \frac{b}{sin(B)} = \frac{c}{sin(C)} = 2 * R$, where R=radius of the circumscribed circle

•Law of Cosines (Generalized Pythagorean Theorem):

$a^2 = b^2 + c^2 - 2 * b * c * cos(A)$

Remarks:

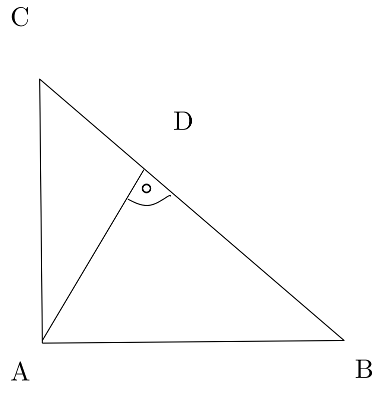
- For A=acute, cos(A) > 0, therefore $-2 * b * c * cos(A)$ is a negative quantity

- For A=obtuse, cos(A) < 0, therefore $-2 * b * c * cos(A)$ is a positive quantity

•Median Theorem: $AD^2 = \frac{2*(AB^2+AC^2)-BC^2}{4}$

•Angle Bisector Theorem: $\frac{BM}{MC} = \frac{AB}{AC}$

Right Triangle Let ABC be a right triangle, A= 90° , AD ⊥ BC



• Formulas for sin, cos, tan, cot

• Values of sin, cos, tan, cot for 30, 60, 45 degrees

• B=90-C, therefore sin(B)=cos(C), cos(B)=sin(C), tan(B)=cot(C), cot(B)=tan(C), etc.

• Pythagorean Theorem: $BC^2 = AB^2 + AC^2$

• Altitude Theorem: $AD^2 = DB * DC$

• Leg Theorem: $AB^2 = BC * DB$, respectively $AC^2 = BC * DC$

- The leg opposite the 30-degree angle = half of the hypotenuse
- S= $\frac{leg*leg}{2}$

Polygons That Can Be Easily Decomposed into Triangles:

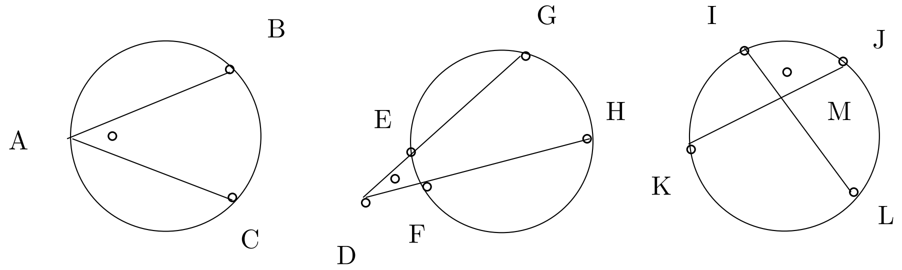
- Square
- Rectangle
- Parallelogram
- Trapezoid
- Rhombus

A useful exercise is the decomposition of the aforementioned figures into triangles and obtaining the formulas for perimeter and area, based on the formulas presented for triangles.

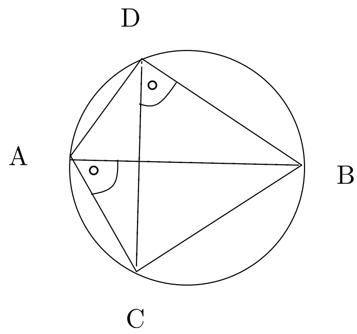
Circle

- Diameter=$2 * R$. Perimeter P=$2 * \pi * R$. Area S=$\pi * R^2$.
- Angle:

a) on the circle: $\hat{A} = \widehat{CAB} = \frac{BC}{2}$
b) exterior: $\hat{D} = \widehat{EDF} = \frac{GH-EF}{2}$
c) interior: $\hat{M} = \widehat{KML} = \frac{IJ+KL}{2}$ according to the figures:



Cyclic Quadrilateral Let ABCD be a cyclic quadrilateral:



In a cyclic quadrilateral the following two properties are valid:

• The sum of opposite angles is 180 degrees, that is $\hat{A} + \hat{B} = 180$, respectively $\hat{C} + \hat{D} = 180$.

• The angle formed by one diagonal with one of the sides is equal to the angle formed by the other diagonal with the opposite side, that is for example $\widehat{CAB} = \widehat{CDB}$.

SHEET 2: Trigonometric Formulas:

$$sin(x) = \frac{opposite \quad leg}{hypotenuse}$$

$$cos(x) = \frac{adjacent \quad leg}{hypotenuse}$$

$$tg(x) = \frac{opposite \quad leg}{adjacent \quad leg}$$

$$ctg(x) = \frac{adjacent \quad leg}{opposite \quad leg}$$

$$tg(x) = \frac{sin(x)}{cos(x)}$$

$$ctg(x) = \frac{cos(x)}{sin(x)}$$

$$tg(x) = \frac{1}{ctg(x)}$$

$$ctg(x) = \frac{1}{tg(x)}$$

$$sin(x) = cos(\frac{\pi}{2} - x) \qquad cos(x) = sin(\frac{\pi}{2} - x) \qquad tg(x) = ctg(\frac{\pi}{2} - x) \qquad ctg(x) = tg(\frac{\pi}{2} - x)$$

$$Fundamental \ formula \ of \ trigonometry:$$

$$sin^2 x + cos^2 x = 1$$

Useful: $cos^2(x) = \frac{1}{1+tg^2(x)}$ $\qquad$ $sin^2(x) = \frac{tg^2(x)}{1+tg^2(x)}$

If we denote $tg\frac{x}{2} = t$, then sin(x), cos(x), tg(x), ctg(x) can be expressed in terms of $tg\frac{x}{2}$ as follows:

$sin(x) = \frac{2t}{1+t^2}$ $\quad$ $cos(x) = \frac{1-t^2}{1+t^2}$ $\quad$ $tg(x) = \frac{sin(x)}{cos(x)} = \frac{2t}{1-t^2}$ $\quad$ $ctg(x) = \frac{cos(x)}{sin(x)} = \frac{1-t^2}{2t}$

$$sin(a+b) = sin(a) * cos(b) + cos(a) * sin(b)$$

$$sin(a-b) = sin(a) * cos(b) - cos(a) * sin(b)$$

$$cos(a+b) = cos(a) * cos(b) - sin(a) * sin(b)$$

$$cos(a-b) = cos(a) * cos(b) + sin(a) * sin(b)$$

$$sin(2a) = 2 * sin(a) * cos(a)$$

$$cos(2a) = cos^2(a) - sin^2(a)$$

$$cos(2a) = 2 * cos^2(a) - 1$$

$$cos(2a) = 1 - 2 * sin^2 a$$

$$sin(3a) = sin(a+2a) = etc...$$

$$cos(3a) = cos(a+2a) = etc...$$

$$sin(\frac{a}{2}) = +-\sqrt{\frac{1-cos(a)}{2}}$$

$$cos(\frac{a}{2}) = +-\sqrt{\frac{1+cos(a)}{2}}$$

$$tg(\frac{a}{2}) = +-\sqrt{\frac{1-cos(a)}{1+cos(a)}}$$

$$ctg(\frac{a}{2}) = +-\sqrt{\frac{1+cos(a)}{1-cos(a)}}$$

$$tg(a+b) = \frac{tg(a)+tg(b)}{1-tg(a)*tg(b)}$$

$$tg(a-b) = \frac{tg(a)-tg(b)}{1+tg(a)*tg(b)}$$

$$ctg(a+b) = \frac{ctg(a)*ctg(b)-1}{ctg(a)+ctg(b)}$$

$$ctg(a-b) = \frac{1+ctg(a)*ctg(b)}{ctg(b)-ctg(a)}$$

$$tg(a+b+c) = tg([a+b]+c) = \frac{tg[a+b]+tg(c)}{1-tg[a+b]*tg(c)} = etc...$$

$$ctg(a+b+c) = ctg([a+b]+c) = \frac{ctg[a+b]*ctg(c)-1}{ctg[a+b]+ctg(c)} = etc...$$

$$sin(a)*cos(b) = \frac{sin(a+b)+sin(a-b)}{2}$$

$$cos(a)*cos(b) = \frac{cos(a+b)+cos(a-b)}{2}$$

$$cos(a)*sin(b) = \frac{sin(a+b)-sin(a-b)}{2}$$

$$sin(a)*sin(b) = \frac{cos(a-b)-cos(a+b)}{2}$$

$$sin(a)+sin(b) = 2*sin(\frac{a+b}{2})*cos(\frac{a-b}{2})$$

$$sin(a)-sin(b) = 2*cos(\frac{a+b}{2})*sin(\frac{a-b}{2})$$

$$cos(a)+cos(b) = 2*cos(\frac{a+b}{2})*cos(\frac{a-b}{2})$$

$$cos(a)-cos(b) = -2*sin(\frac{a+b}{2})*sin(\frac{a-b}{2})$$

The Trigonometric Circle

Write sin(x), cos(x), tg(x), ctg(x) for x=$30°, 45°, 60°$ and respectively $0°, 90°, 180°, 260°, (360)°$

SHEET 3: The Trigonometric Circle

Definition:
The trigonometric circle is a circle with radius R equal to 1 (equal to the unit)

Consequence 1: Expression of Angles in Radians

- Circle length = 2π R = 2π*1= 2π
- Circle length = traversing the entire circle, that is 360 degrees
- Therefore Circle length = 2π = 360
- Consequently: Since 360 degrees = 2π, therefore 180 degrees = π, it follows that 90 degrees = $\frac{\pi}{2}$ and 270 degrees = $\frac{2\pi}{3}$

For example we are interested in how many radians 45 degrees mean.
We start from:

180 degrees .... π

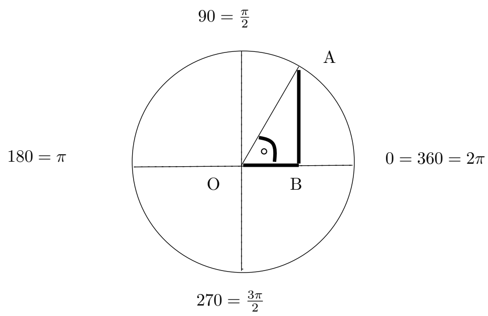
45 degrees (for example)....... x

We apply the rule of three to compute x.

Consequence 2: Finding sin, cos, tg, ctg for 0, 90, 180, 270, 360 degrees Consider the trigonometric circle below, therefore radius R=1.

sin(x)=$\frac{AB}{OA}$= $\frac{AB}{R=1}$ = AB, therefore the segment **AB represents sin(x)**

cos(x)=$\frac{OB}{OA}$= $\frac{OB}{R=1}$ =OB, therefore the segment **OB represents cos(x)**



1) For x=0 degrees, AB becomes 0 and OB becomes 1 (equal to radius R=1), therefore:
sin(0)=0, cos(0)=1, tg(0) = $\frac{0}{1}$=0, ctg(0)=$\frac{1}{0}$ =∞

2) For x=90 degrees, AB becomes 1 (equal to R=1) and OB becomes 0,therefore:
sin(90)=1, cos(90)=0, tg(90) = $\frac{1}{0}$=∞, ctg(90)=$\frac{0}{1}$ =0

3) For x=180 degrees, AB becomes 0 and OB becomes -1 (equal to radius R=1 in negative direction),therefore:
sin(180)=0, cos(180)=-1, tg(180) = $\frac{0}{-1}$=0, ctg(180)=$\frac{-1}{0}$ =-∞

4) For x=270 degrees, AB becomes -1 (equal to R=1 in negative direction) and OB becomes 0,therefore:
sin(270)=-1, cos(270)=0, tg(270) = $\frac{-1}{0}$=-∞, ctg(270)=$\frac{0}{-1}$ =0

Consequence 3: Values of sin, cos, tg, ctg for angles greater than 360 degrees

We consider known from middle school sin(x), cos(x), tg(x), ctg(x) for x=30, 60, 45 degrees:

For x=30: Since sin(x)=$\frac{1}{2}$ and cos(x)=$\frac{\sqrt{3}}{2}$, it follows that

tg=$\frac{sin(x)}{cos(x)}$ = $\frac{1}{\sqrt{3}}$ = $\frac{\sqrt{3}}{3}$ and ctg=$\frac{cos(x)}{sin(x)}$ =$\frac{\sqrt{3}}{1}$=$\sqrt{3}$

For x=60, since 60=90-30, sin(60)=cos(30), cos(60)=sin(30), tg(60)=ctg(30), ctg(60)= tg(30), that is for x=60:
sin(x)=$\frac{\sqrt{3}}{2}$, cos(x)=$\frac{1}{2}$, it follows that tg=$\frac{sin(x)}{cos(x)}$ = =$\sqrt{3}$ and ctg=$\frac{cos(x)}{sin(x)}$ =$\frac{1}{\sqrt{3}}$ = $\frac{\sqrt{3}}{3}$

For x=45, since the right triangle is isosceles (equal legs),
tg(x)=$\frac{opposite\ leg}{adjacent\ leg}$=1,
ctg(x)=$\frac{1}{tg(x)}$=1, the legs being equal and the hypotenuse being the same, sin(x)=cos(x) namely equal to $\frac{\sqrt{2}}{2}$

That is for x=45:
sin(x)=$\frac{\sqrt{2}}{2}$ cos(x)=$\frac{\sqrt{2}}{2}$, tg(x)=1 and ctg(x)=1

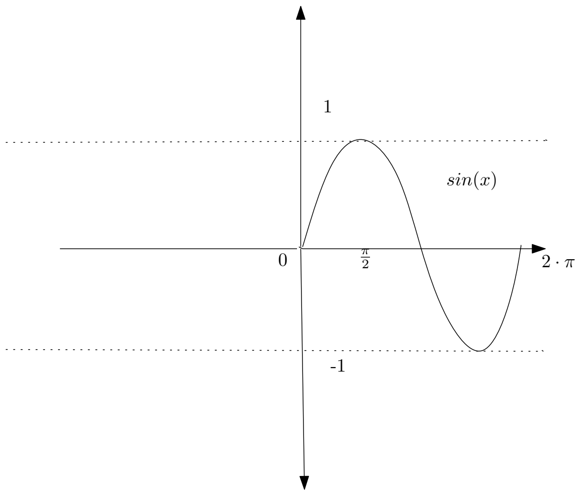
Example 1: Find sin(750)

Since 360 degrees means **one complete rotation,** , 750 degrees means 720(=360+360= two rotations and we arrive back at 0 degrees)+30.
Therefore sin(720) means that point A is in the same position as for 30 degrees.
That is sin(720)=sin(30)=$\frac{1}{2}$.
Identically for cos(x), tg(x), ctg(x), that is cos(750)=cos(30), tg(750)=tg(30), ctg(750)=ctg(30).

Example 2: Find sin($\frac{13\pi}{4}$) First we transform $\frac{13\pi}{4}$ into degrees to judge more easily and obtain 1125 degrees, that is 1125=3* 360+45, meaning we have 3 complete rotations and then another 45 degrees. Therefore sin($\frac{13\pi}{4}$)= sin(45)= $\frac{\sqrt{2}}{2}$.

SHEET 4: Graphs of Trigonometric Functions

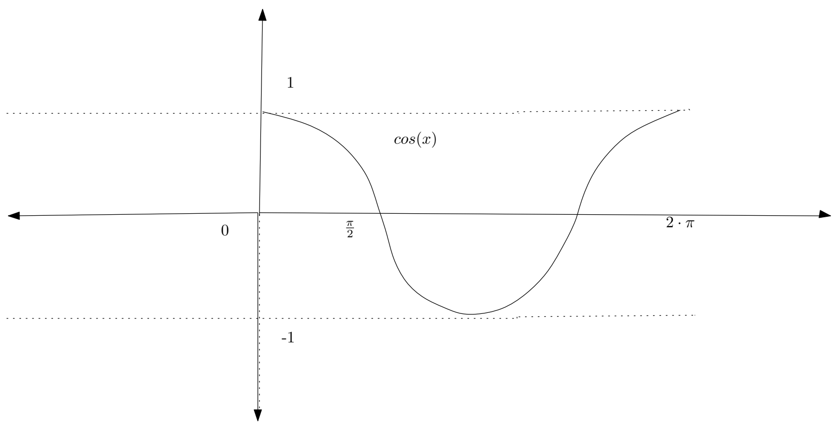
$\sin(x) : \mathbf{R} \to [-1, +1]$



The function sin(x) has period $2 \cdot \pi$, is odd, is defined on R, takes values between -1 and +1 and is bijective for x $\in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Therefore the bijective restriction sin(x) : $[-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, +1]$ admits an inverse function, arcsin(x) : $[-1, +1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$, whose graph is symmetric with respect to sin(x) with respect to the first bisector, y=x.

$\cos(x) : \mathbf{R} \to [-1, +1]$



The function cos(x) has period $2 \cdot \pi$, is even, is defined on R, takes values between -1 and +1 and is bijective for x $\in [0, \pi]$.

Therefore the bijective restriction cos(x) : $[0, \pi] \to [-1, +1]$ admits an inverse function, arccos(x) : $[-1, +1] \to [0, \pi]$, whose graph is symmetric with respect to cos(x) with respect to the first bisector, y=x.

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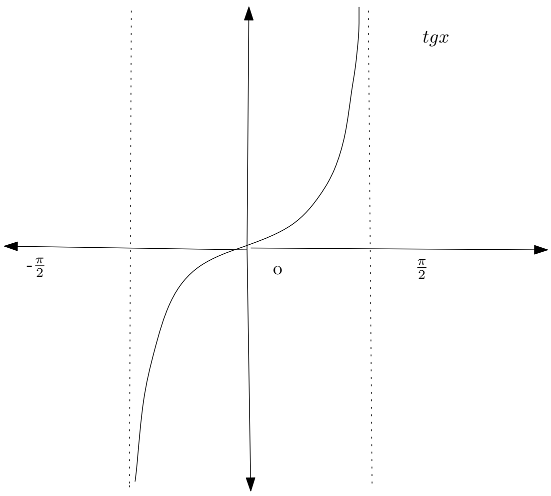
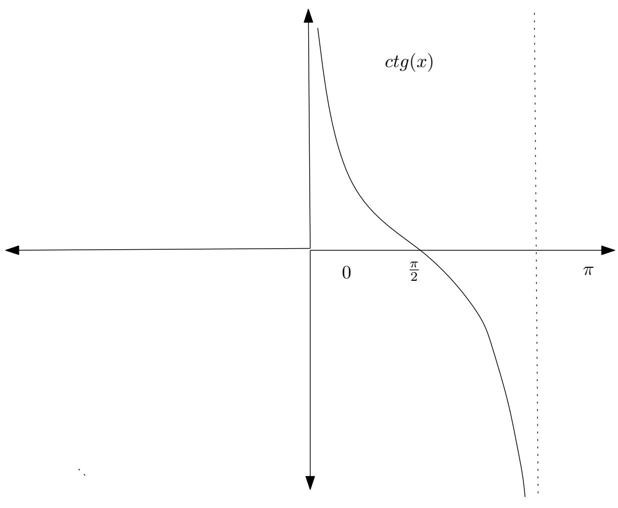
tg(x) : R $\setminus$ { $\frac{\pi}{2}+k\cdot\pi$ } $\to$ R



The function tg(x) has period π, is odd, is defined on R $\setminus$ { $\frac{\pi}{2}+k\cdot\pi$ }, takes values between -∞ and +∞ and is bijective for x $\in$ $(-\frac{\pi}{2},+\frac{\pi}{2})$.

Therefore the bijective restriction tg(x) : $(-\frac{\pi}{2},+\frac{\pi}{2})\to(-\infty,+\infty)$ admits an inverse function, arctg(x) : $(-\infty,+\infty)\to(-\frac{\pi}{2},+\frac{\pi}{2})$, whose graph is symmetric with respect to tg(x) with respect to the first bisector, y=x.

ctg(x) : R $\setminus$ { $\pi+k\cdot\pi$ } $\to$ R



The function ctg(x) has period π, is odd, is defined on R $\setminus$ { $\pi+k\cdot\pi$ }, takes values between -∞ and +∞ and is bijective for x $\in$ $(0,\pi)$.

Therefore the bijective restriction ctg(x) : $(0,\pi)\to(-\infty,+\infty)$ admits an inverse function, arctg(x) : $(-\infty,+\infty)\to(0,\pi)$, whose graph is symmetric with respect to ctg(x) with respect to the first bisector, y=x.

SHEET 5: Trigonometric Equations and Inequalities

1. Elementary Trigonometric Equations

The following are considered known:

- The values for sin(x), cos(x), tg(x), ctg(x) are considered known, where x=30°, 45°, 60° and for 90°, 180°, 270°, as well as the method of transforming from degrees to radians.
- The trigonometric circle is considered known
- The graphs of the functions sin(x), cos(x), tg(x), ctg(x) are considered known
- It is considered known that sin(x), tg(x), ctg(x) are odd and cos(x) is even. It is also known that even functions are symmetric with respect to the y-axis, and odd functions are symmetric with respect to the origin
- It is considered known that sin(x) and cos(x) have period $2 \cdot \pi$ and tg(x) and ctg(x) have period π

An **elementary trigonometric equation** is considered to be an equation that has one of the following forms:

sin(x)=a , where a $\in$ [-1,1]
cos(x)=b , where b $\in$ [-1,1]
tg(x) =c , where c $\in$ R
ctg(x)=d , where d $\in$ R

The trap consists in the fact that solving these equations appears very simple, but in fact is complicated. For example for sin(x)=1/2, the first temptation is to quickly give the answer x= 30°. The answer is correct but incomplete, in the sense that there exist an infinity of solutions, namely: 30°, 150° (= 180-30) and for each of them, $+2\pi k$, where k $\in$ Z namely for k>0 means rotations in the trigonometric direction and for k<0 means rotations in the inverse trigonometric direction. Writing the result in radians, the complete solution for the equation sin(x)=1/2 is x$\in$\{ $\frac{\pi}{6}+2k\pi$\} U \{ $\frac{5\cdot\pi}{6}+2k\pi$\}

To avoid performing the previous reasoning each time, it is efficient to memorize the following formulas:

sin(x)=a	x=$(-1)^k\cdot$ arcsin(a) + kπ, for a $\in$ [0,1]
sin(x)=a	x=$(-1)^{k+1}\cdot$ arcsin$\|a\|$+kπ, for a $\in$ [-1,0)
cos(x)=b	x=$\pm$ arccos(b)+2kπ, for b $\in$ [0,1]
cos(x)=b	x=$\pm$ arccos$\|b\|$+(2k+1)π, for b $\in$ [-1,0]
tg(x)=c	x=arctg(c)+kπ
ctg(x)=d	x=arcctg(d)+kπ

The formulas for tg(x)=c and ctg(x)=d are easy to remember.
For sin(x)=a and cos(x)=b the formulas are more difficult to memorize, alternatively one can remember the method of deducing them, using the trigonometric circle.

2. Equations of the type sin(x)=sin(y)

For the following types:

sin(x)=sin(y)
cos(x)=cos(y)
sin(x)=cos(y)
tg(x) = tg(y)
ctg(x)=ctg(y)

To arrive at elementary trigonometric equations, it is recommended to transform the initial equation, for example:

For sin(x) = sin(y) we move everything to one side, sin(x)-sin(y)=0, transform into a product and obtain $2cos\frac{x+y}{2}sin\frac{x-y}{2} = 0$, that is $cos\frac{x+y}{2} = 0$ and $sin\frac{x-y}{2} = 0$ meaning we obtained a system of two elementary trigonometric equations which we solve.

For sin(x)=cos(y), we write sin(x)=sin($\frac{\pi}{2}$-y) and then proceed similarly to the previous problem.

3. Elementary Trigonometric Inequalities

An **elementary trigonometric inequality** is considered to be an inequality that has one of the following forms:

sin(x) $>$ a
cos(x) $>$ b
tg(x) $>$ c
ctg(x) $>$ d

The sign can be $>$, $<$, $>=$,$<=$, etc.

To avoid memorizing other formulas, it is recommended to solve these inequalities as follows:

• For $sin(x) > a$ and $cos(x) > b$ we analyze the **trigonometric circle**, taking into account the rotations $2 \cdot k \cdot \pi$, where k $\in$ Z.
• For $tg(x) > c$ and $ctg(x) > d$ we analyze the **graph of the function tg(x), respectively ctg(x)**, keeping in mind that tg(x) and ctg(x) have period π

4. ACTUAL Solution of Trigonometric Equations (respectively Trigonometric Inequalities)

In principle we follow the following steps:

1. We transform the given equation, respectively inequality, until we arrive at an **elementary equation**, respectively elementary inequality. The transformation is done using various techniques and the usual trigonometric formulas. Among these formulas, particularly useful are: the fundamental formula of trigonometry, the expression of sin(x), cos(x), tg(x), ctg(x) in terms of $tg\frac{x}{2}$, the expression of sin(x) and cos(x) in terms of tg(x), the half-angle formulas, which have radicals, in case the statement contains squared functions and we wish to eliminate squares, etc. Mastery of trigonometric formulas is recommended.

2. We solve the **elementary equation**, respectively elementary inequality, according to the methods presented earlier.

SHEET 6: Complex Numbers

A. Algebraic Form of Complex Numbers

Everything begins with the number $i = \sqrt{-1}$

• **Calculations** with **i**:

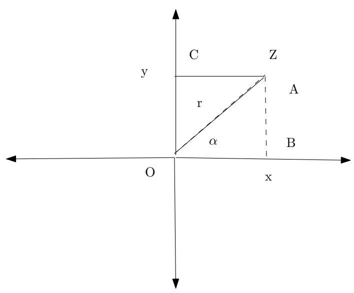
$i = \sqrt{-1}$
$i^2 = (\sqrt{-1})^2 = -1$
$i^3 = i^2 \cdot i = -1 \cdot i = -i$
$i^4 = i^2 \cdot i^2 = -1 \cdot -1 = +1$
It is observed that i^4=1, therefore i to any power that is a multiple of 4 equals 1. This observation is useful for calculating expressions containing i^n.

For example $i^{30} = i^{28} \cdot i^2 = (i^4)^7 \cdot i^2 = 1 \cdot i^2 = -1$

• **New Terms** Let z=x+yi be a complex number (algebraic form).
-**x** is called the *real part* of the complex number and **y** is called the *imaginary part* of the complex number.
-**the conjugate of z** is denoted by $\bar{z}$ and is defined as $\bar{z}$=x-yi

B. Trigonometric Form of Complex Numbers

Let z=x+yi be a complex number (algebraic form).
Consider the following graphical representation:



We can write:
$sin(\alpha) = \frac{y}{r}$ therefore y=r·$sin(\alpha)$, respectively

$cos(\alpha) = \frac{x}{r}$ therefore x=r·$cos(\alpha)$

z=x+yi= r·$cos(\alpha)$+r·$sin(\alpha)$·i= $r[cos(\alpha) + i \cdot sin(\alpha)]$

• The form z=x+yi is called the *algebraic form of the complex number.* **x** is called the *real part* of the complex number and **y** is called the *imaginary part* of the complex number.

• The form z=$r[cos(\alpha) + i \cdot sin(\alpha)]$ is called the *trigonometric form of the complex number.* **r** is called the *modulus* of the complex number and α is called the *argument* of the complex number.

C. Writing a Complex Number in Trigonometric Form

Given a complex number z=x+yi. Write the number in trigonometric form, that is z=r(cosα +i sinα)

1. Determine the quadrant, based on the signs of x and y
2. Calculate $r = \sqrt{x^2 + y^2}$
3. Calculate $tg\alpha^* = |\frac{y}{x}|$ it follows that α^*= arctg$|\frac{y}{x}|$, always α^* being an angle from the first quadrant, because $\frac{y}{x}$ is taken in absolute value.
4. Calculate α based on α^* and based on the quadrant determined in step 1, as follows:

For quadrant 1, take $\alpha = \alpha^*$
For quadrant 2, take $\alpha = \pi - \alpha^*$
For quadrant 3, take $\alpha = \pi + \alpha^*$
For quadrant 4, take $\alpha = 2\pi - \alpha^*$

5. Write the complex number in trigonometric form, that is z=r(cosα +i sinα)

D. Useful Formulas for Complex Numbers in Trigonometric Form

Let $z_1=r_1(\cos\alpha_1 + i \sin\alpha_1)$, $z_2=r_2(\cos\alpha_2 + i \sin\alpha_2)$, $z=r(\cos\alpha + i \sin\alpha)$,

(1) *Multiplication:* $z_1 \cdot z_2 = r_1 \cdot r_2[cos(\alpha_1 + \alpha_2) + i \cdot sin(\alpha_1 + \alpha_2)]$

(2) *Division:* $\frac{z_1}{z_2} = \frac{r_1}{r_2}[cos(\alpha_1 - \alpha_2) + i \cdot sin(\alpha_1 - \alpha_2)]$

Since $z^n = z \cdot z \cdot z.... \cdot z = r \cdot r \cdot ...r[cos(\alpha + \alpha + \cdot\cdot + \alpha) + i \cdot sin(\alpha + \alpha \cdot\cdot + \alpha)]$

we obtain:

(3) *Power:* $z^n = r^n[cos(n \cdot \alpha) + i \cdot cos(n \cdot \alpha)]$

(4) *Radical:* $\sqrt[n]{z} = z^{\frac{1}{n}} = \sqrt[n]{r} \cdot (cos\frac{\alpha+2k\pi}{n} + i \cdot sin\frac{\alpha+2k\pi}{n})$

SHEET 7: Algebraic Formulas:

Formulas for $(a \underline{+} b)^n$

$$(a+b)^2 = a^2 + 2*a*b + b^2$$

$$(a-b)^2 = a^2 - 2*a*b + b^2$$

$$(a+b)^3 = a^3 + 3*a^2*b + 3*a*b^2 + b^3$$

$$(a-b)^3 = a^3 - 3*a^2*b + 3*a*b^2 - b^3$$

$$(a-b)^4 = (a-b)^3*(a-b) \qquad etc...$$

Formulas for $a^n \underline{+} b^n$

$$a^2 - b^2 = (a-b)*(a+b)$$

$$a^3 - b^3 = (a-b)*(a^2 + a*b + b^2)$$

$$a^3 + b^3 = (a+b)*(a^2 - a*b + b^2)$$

$$a^n - b^n = (a-b)*(a^{n-1} + a^{n-2}*b^1 + a^{n-3}*b^2 + ... + b^{n-1})$$

$$a^n + b^n = (a+b)*(a^{n-1} - a^{n-2}*b^1 + a^{n-3}*b^2 - .... + b^{n-1})$$

Sums

$S_1 = 1+2+3+\cdots+n = \frac{n*(n+1)}{2}$ $\qquad$ S_1 is also denoted by $\sum\limits_{k=1}^{n} k$

$S_2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n*(n+1)*(2*n+1)}{6}$ S_2 is also denoted by $\sum\limits_{k=1}^{n} k^2$

$S_3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = S_1^2 = (\frac{n*(n+1)}{2})^2$ S_3 is also denoted by $\sum\limits_{k=1}^{n} k^3$

Example: Calculate the sum $S = 1*2 + 2*3 + ... + n*(n+1)$

Solution: S=$\sum\limits_{k=1}^{n} k*(k+1) = \sum\limits_{k=1}^{n} k^2 + \sum\limits_{k=1}^{n} = S_2 + S_1 = \frac{n*(n+1)*(2*n+1)}{6} + \frac{n*(n+1)}{2}$

Powers and Radicals

$$(a * b)^m = a^m * b^m$$

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

$$(a^m)^n = a^{m*n}$$

$$\frac{a^m}{a^n} = a^{m-n}$$

$$a^0 = 1$$

$$a^{-m} = \frac{1}{a^m}$$

$$\sqrt[m]{a^n} = a^{\frac{n}{m}}$$

Formula for Compound Radicals:
$\sqrt{A \pm \sqrt{B}} = \sqrt{\frac{A+C}{2}} \pm \sqrt{\frac{A-C}{2}}$, where $C = +\sqrt{A^2 - B}$.

Example: Calculate the expression E=$\sqrt{6 - \sqrt{11}}$.

Solution: I calculate C=$\sqrt{6^2 - 11} = \sqrt{25} = 5$, therefore E=$\sqrt{\frac{6+5}{2}} - \sqrt{\frac{6-5}{2}} = \sqrt{\frac{11}{2}} - \sqrt{\frac{1}{2}}$

SHEET 8: Quadratic Equation and Function

Second Degree Equation

$$a * x^2 + b * x + c = 0, \quad where \quad a \neq 0$$

• Roots $x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2*a}$ where $\Delta = \sqrt{b^2 - 4 * a * c}$

• For $\Delta > 0$ there exist two real roots, x_1 *and* x_2 , with $x_1 \neq x_2$

• For $\Delta = 0$ there exist two real, equal and coinciding roots, $x_1 = x_2 = \frac{-b}{2*a}$

• For $\Delta < 0$ there do not exist real roots, or in other words, there exist two imaginary roots x_1 *and* x_2

• Viète's Relations: $S = x_1 + x_2 = \frac{-b}{a}$ $P = x_1 * x_2 = \frac{c}{a}$

• Finding the equation if the sum and product of the roots are known (for example we know that S=10 and P=20).
The equation is: $x^2 - S * x + P = 0$
For the previous example we obtain the equation: $x^2 - 10 * x + 20 = 0$

Second Degree Function

$$y = a * x^2 + b * x + c = 0, \quad where \quad a \neq 0$$

• The graph of the second degree function is a parabola
• Sign of the second degree function:

Rule: *Between roots opposite sign to a, outside roots the sign of a, at the roots equal to zero.*

The application of the rule depends on Δ as follows:
- For $\Delta > 0$ it is exactly according to the rule.
- For $\Delta = 0$, there does NOT exist a zone between roots, therefore the function will be equal to zero at the roots, otherwise it will have the sign of a.
- For $\Delta < 0$, there exist neither roots nor the zone between roots, therefore the function will always have the sign of a.

• **Graphical representation of the second degree function.** Follow the steps:

1) If a > 0, the function has a minimum and if a < 0 it has a maximum.

2)For minimum and maximum the term "Vertex" or "extremum" is also used. This is a point let's say $V(x_v, y_v)$. It represents the lowest point of the graph in the case of the minimum, respectively the highest point in the case of the maximum. Calculate the coordinates of the vertex $V(x_v, y_v)$ with the formula $x_v = \frac{-b}{2*a}$ and $y_v = \frac{-\Delta}{4*a}$, so practically calculate the vertex $V(\frac{-b}{2*a}, \frac{-\Delta}{4*a})$.

3) Find the intersection with the Ox axis, making y=0 in the expression $y = a * x^2 + b * x + c$, that is practically solve the equation $a * x^2 + b * x + c = 0$. The values obtained for x represent the intersections with Ox. For example if we obtain two values x_1 and x_2, the intersection points with Ox will be $(x_1, 0)$ and $(x_2, 0)$.

4) Find the intersection with the Oy axis, making x=0 in the expression $y = a * x^2 + b * x + c$, that is we obtain $y = a * 0^2 + b * 0 + c$,

meaning always we actually obtain y=c. The point (0,c) represents the intersection of the graph with Oy.

5) Draw the graph using the previous data, in the following order: Draw the XOY axes, draw the vertex V, draw the form of the vertex (minimum or maximum), draw the intersections with Ox and the intersection with Oy and connect the points obtained. Keep in mind that the graph is symmetric with respect to the vertex.

- **Monotonicity:**

- The function is decreasing from $-\infty$ up to x_{vertex} and increasing thereafter, from x_{vertex} up to $+\infty$ (if the Vertex is a minimum, that is for $a > 0$). Respectively,

- The function is increasing from $-\infty$ up to x_{vertex} and decreasing thereafter, from x_{vertex} up to $+\infty$ (if the Vertex is a maximum, that is for $a < 0$).

SHEET 9: Sign of the Quadratic Function

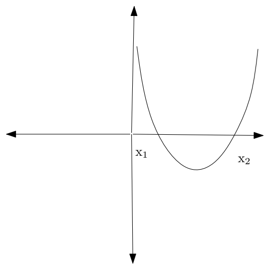
y=ax^2+bx+c, $a \neq$0

The graph of the second degree function is a parabola. If a>0, the graph of the function has a minimum, and if a<0, the graph of the function has a maximum. The minimum, respectively maximum is called the vertex, respectively extreme point. The vertex has coordinates V($\frac{-b}{2a}, \frac{-\Delta}{4a}$)

There are three different situations, depending on whether $\Delta > 0$ or $\Delta = 0$ or $\Delta < 0$

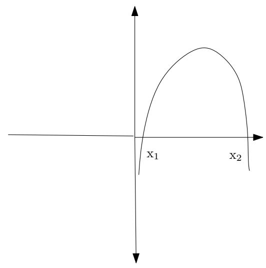
1. For $\Delta > 0$, there exist two real, different roots, $x_1 \neq x_2$, $x_1, x_2 \in R$.

The graph can be of the following type:



$\Delta > 0$ *si* $a > 0$

x	$-\infty$	x_1		x_2		$+\infty$
y=ax^2+bx+c	+ + +	0	- - -	0	+ + +	



$\Delta > 0$ *si* $a < 0$

x	$-\infty$	x_1		x_2		$+\infty$
y=ax^2+bx+x	+ + +	0	- - -	0	+ + +	

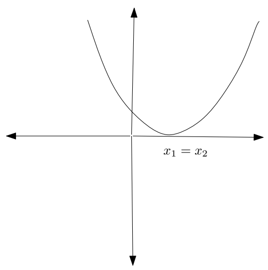
From the analysis of the graphs the following observations can be drawn:

• Between roots the second degree function has a sign opposite to a, and outside the roots it has the sign of a. At the roots, the function equals zero.

• For $\Delta > 0$, the function does not have a constant sign, being positive on certain intervals and negative on others.

2. For $\Delta = 0$, there exist two real, equal roots, $x_1 = x_2, x_1, x_2 \in R$.

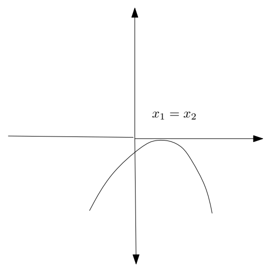
The graph can be of the following type:



$\Delta = 0$ *si* $a > 0$

x	$-\infty$	$x_1 = x_2$		$+\infty$
y=ax^2+bx+c	+ + +	0	+ + +	

y >=0



$\Delta = 0$ *si* $a < 0$

x	$-\infty$	$x_1 = x_2$		$+\infty$
y=ax^2+bx+x	- - -	0	- - -	

y<= 0

From the analysis of the graphs the following observations can be drawn:

• The interval between roots *has disappeared, the roots being equal, respectively coinciding*, therefore the second degree function has the value zero at the roots and otherwise the sign of a.

• For $\Delta = 0$, the function **does not change sign** being positive or equal to zero for $a > 0$, respectively negative or equal to zero for $a < 0$.

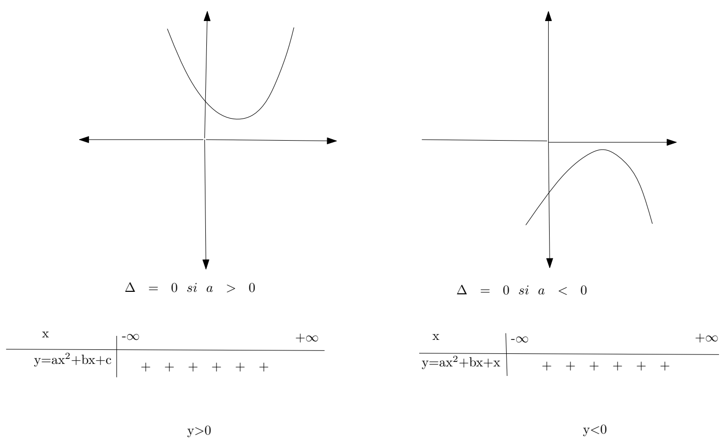
• If in a problem it is necessary to impose the condition for $y >= 0$, we impose the conditions $\Delta = 0$ and $a > 0$.

• If in a problem it is necessary to impose the condition for $y <= 0$, we impose the conditions $\Delta = 0$ and $a < 0$.

---

3. For $\Delta < 0$, there do not exist real roots

The graph can be of the following type:



From the analysis of the graphs the following observations can be drawn:

• The second degree function has the sign of a.

• For $\Delta < 0$, the function **does not change sign** being positive for $a > 0$, respectively negative for $a < 0$.

• If in a problem it is necessary to impose the condition for $y > 0$(strict), we impose the conditions $\Delta < 0$ and $a > 0$.

• If in a problem it is necessary to impose the condition for $y < 0$(strict), we impose the conditions $\Delta < 0$ and $a < 0$.

SHEET 10: Injective, Surjective, Bijective Functions

Injectivity

Definition Let f: A $\rightarrow$ B be a function. The function f is injective if for $x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2)$, for any $x_1, x_2 \in A$

Proving injectivity can be done by several methods, the choice of method depending on the type of function analyzed. For example:

1. If f(x) can be **represented graphically**, for f(x) to be injective, **any line parallel to Ox, drawn through B, must intersect the graph of the function at one point or none.** In other words, any line parallel to Ox through the codomain, must intersect the graph of the function at most at one point.

2. If f(x) can be **represented as a diagram**, usually for problems where the domain of definition A is a finite set, the function is not injective if there exist values in the codomain to which more than one element from the domain of definition corresponds.

3. If f(x) cannot be represented either graphically or as a diagram, one can try **by algebraic calculation**, that is we consider $x_1 \neq x_2$ and try to show that $f(x_1) \neq f(x_2)$. Practically we start from the difference $f(x_1) - f(x_2)$ and through various calculations we try to show that this difference is different from zero. Alternatively we can start from the ratio $\frac{f(x_1)}{f(x_2)}$ and try to demonstrate that this ratio is different from 1.

4. Injectivity can also be verified using material from **mathematical analysis**, studied in the 11th grade. Practically, for the function f(x) we calculate the derivative f'(x) and if we show that f'(x) is always positive on the domain of definition, it means that the function f(x) is increasing. If the function is increasing, it automatically means that the function is injective, because $x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2)$.

Similarly, if instead we obtain f'(x) always negative on the domain of definition, it means that the function f(x) is decreasing. If the function is decreasing, it automatically means that the function is injective, because $x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2)$.

It is observed that we used the property that a monotonic function (regardless of whether it is increasing or decreasing) is injective. The study of the monotonicity of the function was performed through the study of the sign of the derivative of the function. The fact that if f'>0 means that f=increasing, respectively if f'<0 means that f=decreasing, represents consequences of Lagrange's theorem.

Surjectivity

Definition Let f: A $\rightarrow$ B be a function. The function f is surjective if for y $\in$ B, there exists x $\in$ A, such that y=f(x).

Proving surjectivity can be done by several methods, the choice of method depending on the type of function analyzed. For example:

1. If f(x) can be **represented graphically**, for f(x) to be surjective, **any line parallel to Ox, drawn through B, must intersect the graph of the function at one point or more.** In other words, any line parallel to Ox through the codomain, must intersect the graph of the function at least at one point.

2. If f(x) can be **represented as a diagram**, usually for problems where the domain of definition A is a finite set, the function is not surjective if there exist values in the codomain to which **no element from the domain of definition corresponds.**

3. If f(x) cannot be represented either graphically or as a diagram, one can try **by algebraic calculation**, that is **denote f(x) by y and then solve for x as a function of y.** From this expression of x as a function of y, **analyze whether for any y from the codomain, there exists a corresponding x from the domain of definition.**

For example for f:R $\to$ R, f(x)=x^2+1, we make y=x^2+1 and solve for x=$\pm\sqrt{y-1}$. For the radical to exist we need y-1 >=0, that is y>=1, that is y $\in$ [1,∞). The codomain being R, means that for y $\in$ (-∞,1) there does not exist a corresponding x $\in$ R such that y=f(x), therefore the function is not surjective.

4. Surjectivity can also be verified using material from **mathematical analysis**, studied in the 11th grade. The notion of continuity of a function is used and Darboux's property. For example for f(x)=2^x+3^x, since the function is continuous, it means it has Darboux's property, therefore it is surjective. We recall **Darboux's property:**

A function has Darboux's property if it transforms an interval into another interval. That is, for f: I $\to$ J, we say that f has Darboux's property if for the domain of definition I being an interval, we obtain through the function f, the codomain J also as an interval. Continuous functions have Darboux's property. Therefore if I=interval, establishing surjectivity is reduced to studying the continuity of the function and invoking the implication of continuity on Darboux's property.

Bijectivity

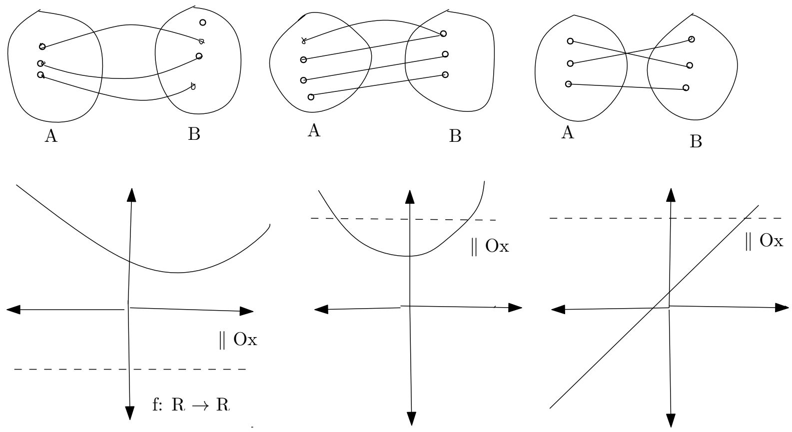
Definition Let f: A $\to$ B be a function. The function f is bijective if it is both injective and surjective.

The study of bijectivity is reduced to verifying, in turn, injectivity and surjectivity according to the methods presented earlier. If the function is both injective and surjective, we conclude that the function is bijective, otherwise we conclude that the function is not bijective.

The following observations can be made:

1. In the case where **graphical representation is used,** for a function to be bijective, that is both injective and surjective, **any line parallel to Ox, drawn through B, must intersect the graph of the function exactly at one point**.

2. In the case where **diagram representation is used,** for a function to be bijective, that is both injective and surjective, in the diagram there must be a 1 to 1 correspondence, that is to any element x $\in$ A there corresponds exactly one element y $\in$ B and vice versa. In other words, there must be a **1 to 1 correspondence**, which is also called **one-to-one correspondence**. **Examples:**



Non surjective function **Non injective function** **Bijective function**

SHEET 11: Arithmetic Progressions. Geometric Progressions

Arithmetic Progressions

• Definition: Arithmetic progression = a sequence $a_1, a_2, ..., a_n$ with the property that each term equals the previous term plus the common difference (denoted r). In other words, $a_2 - a_1 = r$, $a_3 - a_2 = r$, etc. More precisely, $a_n - a_{n-1} = r$ for $n >= 2$. That is an A.P. is a sequence $a_1, a_2, ..., a_n$ with the property that

$$\mathbf{a_n = a_{n-1} + r}$$

This is formula 1. The common difference can be either positive or negative.

It is necessary that n be >= 2, because a_1 has no previous term.

• Formula 2 = Expression of a_n as a function of the first term (a_1) and common difference (r).

Since $a_1 = a_1$
Since $a_2 = a_1 + r$
Since $a_3 = a_2 + r = a_1 + 2 * r$
Since $a_4 = a_3 + r = a_1 + 3 * r$, etc., we obtain the formula:

$$\mathbf{a_n = a_1 + (n - 1) * r}$$

• Formula 3 = Verification if three numbers are in arithmetic progression

To verify if the numbers A, B, C are in A.P., we verify if the condition is satisfied:

$$\mathbf{B = \frac{A + C}{2}}$$

that is if the middle one is the arithmetic mean of the neighbors.

It is natural for it to be so, because A= B - r, C= B+r, therefore $A+C = 2 * B$

• Formula 4 = Sum of an arithmetic progression

For the arithmetic progression $a_1, a_2, ..., a_n$, the sum

$$\mathbf{S = a_1 + a_2 + ... + a_n = \frac{(a_1 + a_n) * n}{2}}$$

It is easily remembered by the formulation $\mathbf{S = \frac{(FirstTerm+LastTerm)*n}{2}}$

The proof is simple, namely:

S=a_1+a_2+...+a_n $= a_1+(a_1+r)+(a_1+2*r)+(a_1+3*r)+....(a_1+(n-1)*r)$
$= (a_1*n+r+2*r+3*r+...(n-1)*r) = n*a_1+r*(1+2+...+(n-1)) =$
$= n * a_1 + r * \frac{(n-1)(n-1+1)}{2} = n * a_1 + \frac{r*n*(n-1)}{2} = \frac{2*a_1+(n-1)*r}{2} * n =$
$\frac{a_1+a_1+(n-1)*r}{2} * n$

Therefore S=$\frac{(a_1+a_n)*n}{2}$

• Formula 5 = Sum of an arithmetic progression, alternative formula

If we express $a_n = a_1 + (n - 1) * r$ in the previous formula, we obtain another formula for S, namely:

$$\mathbf{S = \frac{(2 * a_1 + (n - 1) * r) * n}{2}}$$

Recommendation: If in the problem the first term and last term are known, formula 4 is useful, and if the first term and common difference are known, formula 5 is useful. In both situations we need to know the number of terms of the progression, denoted by n.

Geometric Progressions

• Definition: Geometric progression = a sequence $b_1, b_2, ..., b_n$ with the property that each term equals the previous term multiplied by the common ratio (denoted q). In other words, $b_2 = b_1 * q$, $b_3 = b_2 * q$, etc. More precisely, $b_n = b_{n-1} * q$ for $n >= 2$. That is a G.P. is a sequence $b_1, b_2, ..., b_n$ with the property that

$$\mathbf{b_n = b_{n-1} * q}$$

This is formula 1. The common ratio can be either positive or negative. It is necessary that n be >= 2, because b_1 has no previous term.

• Formula 2 = Expression of b_n as a function of the first term (b_1) and common ratio (q).
Since $b_1 = b_1$
Since $b_2 = b_1 * q$
Since $b_3 = b_2 * q = b_1 * q^2$
Since $b_4 = b_3 * q = b_1 * q^3$, etc., we obtain the formula:

$$\mathbf{b_n = b_1 * q^{n-1}}$$

• Formula 3 = Verification if three numbers are in geometric progression
To verify if the numbers A, B, C are in G.P., we verify if the condition is satisfied:

$$\mathbf{B = \sqrt{A * C}}$$

that is if the middle one is the geometric mean of the neighbors.
It is natural for it to be so, because $A = \frac{B}{q}$, $C = B * q$, therefore $A * C = B^2$, that is $B = \sqrt{A * C}$

• Formula 4 = Sum of a geometric progression
For the geometric progression $b_1, b_2, ..., b_n$, the sum

$$\mathbf{S = b_1 + b_2 + ... + b_n = b_1 * (\frac{q^n - 1}{q - 1})}$$

The proof is simple, namely:
S=$b_1 + b_2 + ... + b_n = b_1 + (b_1 * q) + (b_1 * q^2) + (b_1 * q^3) + ...(b_1 * q^{n-1})$
$= b_1(1 + q + q^2 + q^3 + ... + q^{n-1}) = b_1 * (\frac{q^n - 1}{q - 1})$ Q.E.D.

We used the formula $x^n - 1^n = (x - 1) * (x^{n-1} + x^{n-2} + x^{n-3} + ... + x + 1)$ where in the role of x we used q and we extracted $(1 + x + x^2 + ... + x^{n-1}) = \frac{x^n - 1}{x - 1}$

SHEET 12: Exponential Function

y=a^x, a >0 , a$\neq 1$

There are two different situations, depending on whether a>1 or a∈(0,1)

Justification of the fact that a >0 , a$\neq 1$

The value of **a cannot have** the following values:

-It should not be that a equals 0, because 0 to any power equals 0.

-It should not be that a equals 1, because 1 to any power equals 1.

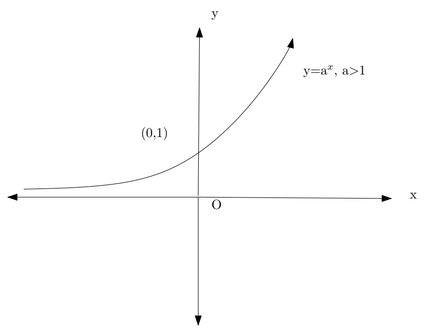
-It should not be that a is negative. For example if we have a=-3 and x=$\frac{1}{2}$, we would obtain a^x = (-3)$^{\frac{1}{2}}$, that is a^x = $\sqrt{-3}$, that is the square root of a negative number, which does not belong to R.

From the values a $\in$ R, if we eliminate a=0, a=1 and a<0, we obtain two permitted intervals, namely a $\in$ (0,1) and a $\in$ (1,∞), or in other words, a>0 and a $\neq$1

Case 1. a>1 For example let y=2^x. We draw the graph of the function by points, giving values to x:

```
 x | ...   -3   -2     -1   0   1   2  3 ....
---|---------------------------------------
 y |       1/8  1/4    1/2  1   2   4  8
```

We obtain the following graph:



From the analysis of the graph the following observations can be drawn:

• The function y=a^x, a> 1 is defined on R and takes values in (0,∞), that is
f: R → (0,∞), therefore the domain of definition is R, and the codomain is (0,∞).
It is observed that any line parallel to the Ox axis, drawn through the codomain intersects the graph of the function at exactly one point, therefore the function is bijective.
Being bijective, it means that it is both surjective and injective.
Due to the fact that it is injective, that is if f(x_1)=f(x_2) it means that x_1=x_2, in the case where we obtain an equation of the form $a^{f(x)}$=$a^{g(x)}$, we can conclude that f(x)=g(x).

• $a^x > 0$ for any x ∈ R.
Therefore if we obtain an equation for example 5^x=-25 it means that the equation has no solutions, because 5^x is always positive.

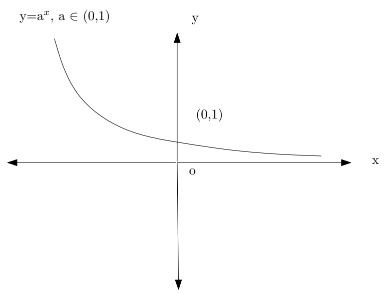
• The function a^x, for a> 1 is **increasing**, that is if $x_1 < x_2$ it means that $f(x_1) < f(x_2)$ and vice versa. If we obtain an inequality, for example 2^x <8, we can conclude that x< 3.

• Since a^0=0, the graph of the function y=a^x passes through the point (0,1) for any a> 1

Case 2. a∈(0,1) For example let y=$(\frac{1}{2})^x$. We draw the graph of the function by points, giving values to x:

x	...	-3	-2	-1	0	1	2	3	....
y		8	4	2	1	1/2	1/4	1/8	

We obtain the following graph:



From the analysis of the graph the following observations can be drawn:

• The function y=a^x, a$\in$ (0,1) is defined on R and takes values in (0,∞), that is
f: R $\to$ (0,∞), therefore the domain of definition is R, and the codomain is (0,∞). It is observed that any line parallel to the Ox axis, drawn through the codomain intersects the graph of the function at exactly one point, therefore the function is bijective. Being bijective, it means that it is both surjective and injective. Due to the fact that it is injective, that is if f(x_1)=f(x_2) it means that x_1=x_2, in the case where we obtain an equation of the form a$^{f(x)}$=a$^{g(x)}$, we can conclude that f(x)=g(x).

• a^x $>$ 0 for any x $\in$ R. Therefore if we obtain an equation for example $(\frac{1}{5})^x$=-$(\frac{1}{25})$ it means that the equation has no solutions, because $(\frac{1}{5})^x$ is always positive.

• The function a^x, for a$\in$ (0,1) is **decreasing**, that is if $x_1 < x_2$ it means that $f(x_1) > f(x_2)$ and vice versa. If we obtain an inequality for example $(\frac{1}{2})^x < \frac{1}{8}$, we can conclude that x$>$ 3.

• Since a^0=0, the graph of the function y=a^x passes through the point (0,1) for any a$\in$ (0,1).

• **Practically for solving exponential equations** we try to arrive through various techniques at an equation of the form a$^{f(x)}$=a$^{g(x)}$, from which we conclude that f(x)=g(x).

• **Practically for solving exponential inequalities** we try to arrive through various techniques at an inequality of the form a$^{f(x)}$ $<$ a$^{g(x)}$, which is interpreted according to the value of a, namely if a$>$ 1 we preserve the sign of the inequality between f(x) and g(x), respectively if a$\in$ (0,1), we reverse the sign of the inequality between f(x) and g(x).

• It is useful to recall the formulas for powers
$(a^m)^n = a^{m+n}$, $\frac{a^m}{a^n} = a^{m-n}$, a^0=1, a^{-m} = $\frac{1}{a^m}$, $\sqrt[m]{a^n} = a^{\frac{n}{m}}$

• **The exponential function** y=a^x : R $\to$ (0,∞) being bijective, admits an inverse function namely **the logarithmic function** y=$\log_a^x$: (0,∞) $\to$ R, whose graph is symmetric with respect to y=a^x with respect to the first bisector, y=x. Of course two situations are generated for $\log_a^x$ namely for a$>$1 and for a$\in$(0,1).

SHEET 13: Logarithms

Formulas:

1) Definition: log_a^X is a number N, with the property that,

$$\text{if } \ log_a^X = N, \ \text{then } X = a^N$$

For example $log_{10}^{100} = 2$ because $100 = 10^2$

2) $log_a^A + log_a^B = log_a^{A*B}$

3) $log_a^A - log_a^B = log_a^{\frac{A}{B}}$

4) $log_a^{A^m} = m * log_a^A$

5) $log_a^a = 1$

6) $log_{a=old}^X = \frac{log_{b=new}^X}{log_{b=new}^{a=old}}$

7) $log_{a^m}^X = \frac{1}{m} * log_a^X$

8) $log_a \sqrt[m]{X^n} = log_a^{X^{\frac{n}{m}}} = \frac{n}{m} * log_a^X$

Domain of definition:

For $log_a A$ it is necessary that

- A >0 → we obtain for example x belongs to interval I_1
- a >0 and a ≠1 → we obtain for example x belongs to interval I_2

The domain of definition is I_1 intersected with I_2

Base of the logarithm:

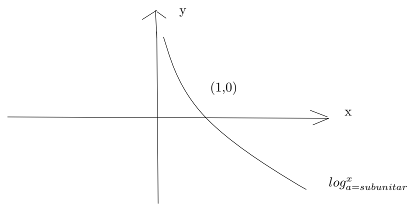
- lg^X means log_{10}^X that is *common logarithm*
- ln^X means log_e^X that is *natural logarithm*, where e=2.7

Subunitary base or superunitary base.

Due to the condition from the *domain of definition*, for log_a^X, since we need $a > 0$ and $a \neq 1$, practically there are two situations for the base of the logarithm:

-CASE 1: a between (0,1). In this case, the logarithm function is decreasing, for example if we know that a is between (0,1) and we obtain in a problem $log_a^A < log_a^B$, we can draw the conclusion that $A > B$.

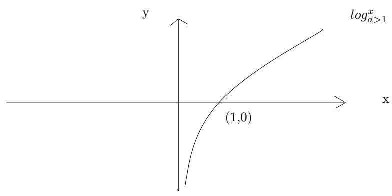
The graph of the logarithm function, for **subunitary** base is the following:



The following can be observed from the graph:

- $log_a^X : R \rightarrow (0,\infty)$
- the function is decreasing
- $log_a^1 = 0$
- the function is positive for X between (0,1)
- the function is zero for X=1
- the function is negative for X>1.

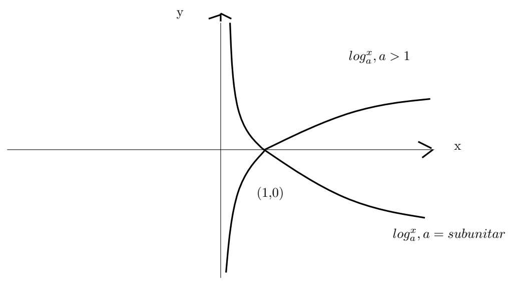
- **CASE 2: a between (1,∞)** In this case, the logarithm function is increasing, for example if we know that a is between (1,∞) and we obtain in a problem $log_a^A < log_a^B$, we can draw the conclusion that $A < B$. The graph of the logarithm function, for **superunitary** base is the following:



The following can be observed from the graph:

- $log_a^X : R \rightarrow (0,\infty)$
- the function is increasing
- $log_a^1 = 0$
- the function is negative for X between (0,1)
- the function is zero for X=1
- the function is positive for X>1.

Both cases can be synthesized in a single graph:



SHEET 14: Combinatorial Analysis

Factorial, Permutations, Arrangements, Combinations

Factorial

• By definition **n factorial** is denoted by **n!** and has the following calculation formula:

$$n! = 1 \cdot 2 \cdot 3 \cdot .... \cdot (n-1) \cdot n$$

• Examples:
a) 3!=$1 \cdot 2 \cdot 3 = 6$
b) 1!=1
c) 0!=1 (surprising at first sight, will be explained later)
• The following formulas result:
n!=$1 \cdot 2 \cdot 3 \cdot .... \cdot (n-1) \cdot n$, therefore it can also be written as follows:
n!=(n-1)!$\cdot n$
n!=(n-2)!$\cdot (n-1) \cdot n$
n!=(n-3)!$\cdot (n-2) \cdot (n-1) \cdot n$, etc.

Permutations

• **Permutations of n elements** is denoted by P_n and has the following calculation formula:

$$P_n = n!$$

therefore $P_n = n! = 1 \cdot 2 \cdot 3 \cdot .... \cdot (n-1) \cdot n$
• Example:
a) P_3 = 3!=$1 \cdot 2 \cdot 3 = 6$
• What does "Permutations of n elements" mean

If we have for example a number of n=3 elements $\{a, b, c\}$, P_3 shows us **in how many ways** we can "permute (= exchange among themselves the 3 elements)", so as to form "teams" that:
a) contain all the n elements (both a and b and c)
b) no element repeats.

Practically we can form the following "teams":
{abc} {acb} {bac} {bca} {cab} {cba}

If we count the "previous teams" we see that there are 6. Indeed 6= $P_3 = 3! = 1 \cdot 2 \cdot 3$. In conclusion with P_n we can find **the number of "teams"** that can be generated, so it tells us **how many teams there are, not which teams these are**.

Arrangements

• **Arrangements of n elements taken k at a time** is denoted by A_n^k and has the following calculation formula:

$A_n^k = n \cdot (n-1) \cdot (n-2) \cdot .... \cdot (n-k+1)$

• Example:
$A_{10}^4 = 10 \cdot 9 ... (10-4+1) = 10 \cdot 9 \cdot 8 \cdot 7 = 5040$

• What A_n^k means:

Example 1: For example if we have 30 students and we want to group them 2 at a time in desks, the number of these groupings is given by A_{30}^2, namely according to the previous formula we can calculate $A_{30}^2 = 30 \cdot 29 = 870$ that is they can be "arranged" in 870 ways.

Example 2: If we have n=3 elements for example the set $\{a, b, c\}$ and we want to "arrange" these elements in groups of 2, we obtain:

{ab} {ac} {bc} {ba} {cb} {ca} We observe that there are 6 groupings, that is $A_3^2 = 3 \cdot 2 = 6$. It is observed that both the grouping $\{ab\}$ and the grouping $\{ba\}$ appear that is **the order of the elements matters.**

Practically A_n^k is similar to P_n with the difference that in the "team" not all the n elements enter but only k (where k $\leq$ n).

• Domain of definition:

For A_n^k we need n, k $\in$ N and n $\geq$ k

One can imagine the absurdity of situations for n $<$ k or for n, k non-natural (for example negative, fractional, etc.)

Combinations

• **Combinations of n elements taken k at a time** is denoted by C_n^k and has the following calculation formula:

$C_n^k = \frac{A_n^k}{P_k}$

• **Practically** The formula $C_n^k = \frac{n!}{k!(n-k)!}$ is used

• Example:
$C_{10}^4 = \frac{10!}{4! \cdot 6!} = \frac{6! \cdot 7 \cdot 8 \cdot 9 \cdot 10}{6! \cdot 1 \cdot 2 \cdot 3 \cdot 4} = 210$

• What C_n^k means:

Example: If we have n=3 elements for example the set $\{a, b, c\}$ and we want to "combine" these elements in groups of 2, we obtain:

{ab} {ac} {bc} We observe that there are 3 groupings, that is $C_3^2 = 3$. It is observed that the grouping $\{ab\}$ appears **but not the grouping** $\{ba\}$ that is **the order of the elements does not matter.**

Practically C_n^k is similar to A_n^k with the difference that in the "team" the order of the elements does not matter, that is {a,b } is considered identical to {b,a }, so it is "counted" only once . It is also observed from the definition formula that $C_n^k \leq A_n^k$

• Domain of definition:

For C_n^k we need n, k $\in$ N and n $\geq$ k, as in the case of arrangements.

1. Newton's Binomial Formula

It is observed that the following formulas have similar structure:

$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, etc.

In general for $(a+b)^n$ the following are observed:

There are n+1 terms, the first term being a^n, and the last b^n. More precisely the first is $a^n b^0$ and the last $a^0 \cdot b^n$, in general we have terms of the form $a^{n-k}b^k$, where k takes values from 0 to n. These terms are each multiplied by a coefficient. These coefficients have the formula C_n^k.

The grouping $(a+b)^n$ being formed of two terms, is called a binomial. After the name of the person who discovered the following formula, this is called **Newton's binomial**:

$$(a+b)^n = C_n^0 \cdot a^{n-0} \cdot b^0 + C_n^1 \cdot a^{n-1} \cdot b^1 + C_n^2 \cdot a^{n-2} \cdot b^2 + \cdots C_n^n \cdot a^{n-n} \cdot b^n$$

The following observations can be made:

- There are n+1 terms, because k takes values between 0 and n.

- The first term is actually equal to a^n, because $C_n^0=1$ and $b^0=1$.

- The last term is actually equal to b^n, because $C_n^n=1$ and $a^0=1$.

- The coefficients C_n^0, C_n^1, $\cdots$ C_n^n are called **binomial coefficients**.

- **Equidistant binomial coefficients are equal**, because it can be easily demonstrated that $C_n^k = C_n^{n-k}$.

- Newton's binomial can be written concisely in the following form:

$$(a+b)^n = \sum_{k=0}^{n} C_n^k \cdot a^{n-k} \cdot b^k$$

therefore as a sum having the general term

$$T_{k+1} = C_n^k \cdot a^{n-k} \cdot b^k$$

It is observed that the first term T_1 has k=0, the second term T_2 has k=1, etc. so indeed the notation of T_{k+1} for a value k is justified.

- If the binomial is a difference instead of a sum:

$$(a-b)^n = (a+[-b])^n = C_n^0 \cdot a^{n-0} \cdot (-b)^0 + C_n^1 \cdot a^{n-1} \cdot (-b)^1 + C_n^2 \cdot a^{n-2} \cdot (-b)^2 + \cdots C_n^n \cdot a^{n-n} \cdot (-b)^n$$

that is there is an alternation of signs starting with + then - and so on.

$$(a-b)^n = (a+[-b])^n = C_n^0 \cdot a^{n-0} \cdot b^0 - C_n^1 \cdot a^{n-1} \cdot b^1 + C_n^2 \cdot a^{n-2} \cdot b^2 + \cdots + (-1)^n \cdot C_n^n \cdot a^{n-n} \cdot (-b)^n$$

- Now we can write Newton's binomial formula in the most general way:

$$(a \pm b)^n = \sum_{k=0}^{n} (-1)^k \cdot C_n^k \cdot a^{n-k} \cdot b^k$$

therefore as a sum having the general term

$$T_{k+1} = (-1)^k \cdot C_n^k \cdot a^{n-k} \cdot b^k$$

---

2. Useful Formulas

• Most problems with Newton's binomial are solved **starting from the formula**

$$T_{k+1} = (-1)^k \cdot C_n^k \cdot a^{n-k} \cdot b^k$$

• The following two binomials are developed with Newton's binomial:

$(1+1)^n = 2^n = C_n^0 + C_n^1 + C_n^2 \cdots C_n^n \qquad (1)$

$(1-1)^n = 0 \ = C_n^0 - C_n^1 - C_n^2 \cdots C_n^n \qquad (2)$

Adding relation (1) to relation (2) we obtain: $C_n^0 + C_n^2 + C_n^4 \cdot \cdot \cdot = 2^{n-1}$

Subtracting relations (1) - (2) we obtain: $C_n^1 + C_n^3 + C_n^5 \cdot \cdot \cdot = 2^{n-1}$

In conclusion **the sum of even binomial coefficients equals the sum of odd binomial coefficients and equals** 2^{n-1}:

$$C_n^0 + C_n^2 + C_n^4 \cdot \cdot \cdot = C_n^1 + C_n^3 + C_n^5 \cdot \cdot \cdot = 2^{n-1}$$

• For problems where it is required to determine the rank of the largest term in the expansion, we start from the ratio $\frac{T_{k+1}}{T_{k+2}}$ which is compared with 1. That is we start from $\frac{T_{k+1}}{T_{k+2}} > 1$ and from this relation we solve for k. Then we interpret k and find the maximum term. For a binomial $(a+b)^n$, the ratio

$$\frac{T_{k+1}}{T_{k+1}} = \frac{n-k}{k+1} \cdot \frac{b}{a}$$

The formula being difficult to remember, it is recommended to demonstrate ad-hoc, starting from the ratio $\frac{T_{k+1}}{T_{k+2}}$ and expressing the terms T_{k+1} and T_{k+2} with the usual formula $T_{k+1} = (-1)^k \cdot C_n^k \cdot a^{n-k} \cdot b^k$

Examples: Find the rank of the largest term in the expansion:

$$a)\ (1+0.1)^{100} \qquad b)\ (\frac{1}{2} + \frac{1}{2})^{100} \qquad c)\ (\frac{3}{4} + \frac{1}{4})^{100}$$

SHEET 16: Polynomials

1. Division of Polynomials

• **Method 1** The classical method of dividing two polynomials f(x) and g(x), according to the method studied in the 8th grade. It has the advantage that the divisor g(x) can be of any degree. It is recommended to perform the verification according to the formula:

Dividend = Divisor × Quotient + Remainder

• **Method 2** Using Horner's scheme for dividing two polynomials f(x) and g(x), where g(x) is of the form (x-a). It has the disadvantage that g(x) must be of degree I. If g(x) is of higher degree, we decompose g(x) into factors of degree I, for example x^2-25 =(x-5)(x-(-5)) and we divide successively f by (x-5) and then the quotient obtained is divided by x-(-5). Horner's scheme has the advantage that it lends itself to computer processing.

2. Divisibility of Polynomials

To determine the *the greatest common divisor* **(gcd)** of two polynomials, the **Euclidean algorithm** is used, namely:

We divide f(x) by g(x) and obtain a quotient and a remainder. Then we divide the dividend by the obtained remainder. We continue performing divisions of the dividend by the remainder, until we obtain remainder=0. **The last non-zero remainder is gcd(f,g).**

Note 1 If *the least common multiple* **lcm(f,g)** is required, we use the property that

$$\mathbf{lcm(f,g) \cdot gcd(f,g) = f \cdot g}$$

Practically, we calculate f ·g and then we calculate gcd(f,g) with the Euclidean algorithm. Then we find lcm(f,g)=$\frac{f \cdot g}{gcd(f,g)}$

Note 2 If it is required to verify whether f and g are **relatively prime**, we use the property that **two polynomials are relatively prime if their gcd=1.**

Practically, we calculate gcd(f,g) with the Euclidean algorithm and if we obtain gcd(f,g)=1 we draw the conclusion that f and g are relatively prime, otherwise we draw the conclusion that they are not relatively prime.

3. Roots of Higher Degree Equations. Multiple Roots.

• A polynomial P_n of degree n, having roots $x_1, x_2, \cdots x_n$ can be written in the form:

$$P_n = (x - x_1) \cdot (x - x_2) \cdots (x - x_n)$$

• If a polynomial P(X) admits root x=a, then P(a)=0 (Bézout's Theorem).

• Multiple roots. An equation can have multiple roots (double, triple, etc.), in general of order k of multiplicity.

For example x=α is a double root if x_1=x_2=α. In this case P(x) appears in the form:

$$P(x)_n = (x - \alpha)^2 \cdot (x - x_3) \cdots (x - x_n)$$

It means that P(x) is exactly divisible by (x-α)^2, that is it is divided for example with Horner's scheme by (x-α) and then the obtained quotient is divided also exactly by (x-α). Another approach to the problem is to divide P(x) by (x-a)^2, that is by x^2-2ax+a^2 by the classical method of polynomial division and impose the condition that the remainder is zero.

The most effective for multiple roots, is to use the following theory:

A polynomial P(x) has root x=α as a multiple root of order k of multiplicity, if:

$$P(\alpha) = 0$$
$$P^{'}(\alpha) = 0$$
$$P^{''}(\alpha) = 0$$
$$\ldots$$
$$P^{k-1}(\alpha) = 0$$
$$P^{k}(\alpha) \neq 0$$

---

4. Viète's Relations

For example let the 3rd degree equation: $ax^3+bx^2+cx+d=0$ having roots x$_1$,x$_2$,x$_3$.

The following sums, called Viète's relations, can be written:

$$\begin{cases} S_1 = & x_1 + x_2 + x_3 & = -\frac{b}{a} \\ S_2 = & x_1 \cdot x_2 + x_1 \cdot x_3 + x_2 \cdot x_3 & = +\frac{c}{a} \\ S_3 = & x_1 \cdot x_2 \cdot x_3 & = -\frac{d}{a} \end{cases}$$

Similarly Viète's relations can be written for any degree n.

If the roots of an equation are known, for example y$_1$,y$_2$,y$_3$ $\cdots$ y$_n$, and we wish to find the equation that has those roots, we calculate Viète's sums S$_1$,S$_1$, $\cdots$ S$_n$, then we write the expression of the equation that has those roots:

$$1 \cdot Y^n - S_1 \cdot Y^{n-1} + S_2 \cdot Y^{n-2} \cdots S_n = 0$$

---

5. Advanced Solutions of Higher Degree Equations

In addition to the classical methods of solving higher degree equations, advanced methods that use derivatives can also be employed, for example:

- Using **Rolle's Sequence**.
- Using the graphical representation of the functions that form the equation.

SHEET 17: Higher Degree Equations

Part I

Type 1. Biquadratic Equations

Example: Solve the equation $x^4 - 6x^2 + 6 = 0$

Idea: Denote $x^2 = y$ and obtain a second degree equation which is solved, then find x_1, x_2, x_3, x_4.

Type 2. Reciprocal Equations of Degree Three

Example: Solve the equation $5x^3 + 31x^2 + 31x^1 + 5 = 0$

Theory 1: A **reciprocal equation** is an equation which has equidistant coefficients that are equal.

Theory 2: Any reciprocal equation of odd degree admits the root x=-1.

Theory 3: A polynomial P_n of degree n, having roots $x_1, x_2, \cdots x_n$ can be written in the form:

$$P_n = (x - x_1) \cdot (x - x_2) \cdots (x - x_n)$$

Theory 4: If we have for a division Dividend, Divisor, Quotient, Remainder, the following relation is correct:

$$Dividend = Quotient \cdot Divisor + Remainder$$

Idea: Let $P_3(x)$ be the expression equal to zero. Since it admits the root x=-1, it means that $P_3(x)$ is divisible by x-(-1) that is by x+1. We perform the division of $P_3(x)$ by (x+1) and obtain $Quotient_2(x)$ and remainder=0. Therefore $P_3(x) = (x + 1) \cdot Quotient_2(x)$. We solve $Quotient_2(x) = 0$ and find the other two roots.

Type 3. Reciprocal Equations of Degree Four

Example: Solve the equation $2x^4 + 7x^3 + 9x^2 + 7x^1 + 2 = 0$

Idea: Divide the equation by x^2, after which denote $(x + \frac{1}{x}) = y$. Express everything in terms of y. Solve the second degree equation in y, then find x_1, x_2, x_3, x_4.

Type 4. Reciprocal Equations of Degree Five

Example: Solve the equation $20x^5 - 81x^4 + 62x^3 + 62x^2 - 81x^1 + 20 = 0$

Idea: Being a reciprocal equation of odd degree, it has the root x=-1. We proceed as in the case of reciprocal equations of degree three and from $P_5(x) = (x + 1)Q_4(x)$, by dividing $P_5(x)$ by (x+1) we obtain $Q_4(x)$ as a reciprocal equation of degree 4, which is solved like any reciprocal equation of degree four.

Type 5. Equations that Admit Root $x = a + b \cdot i$

Example 1: Solve the equation $ax^4 + bx^3 + cx^2 + dx^1 + e = 0$ knowing that it admits the root x=1+i.

Theory: If an equation admits the root x=a+bi, then it also admits the root x=a-bi

Idea: Let for example the polynomial $P_4(x)$ which we know admits the root $x_1 = a + bi$. According to the theory, $x_2 = a - bi$. This means that we can write

$$P_4(x) = (x - x_1) \cdot (x - x_2) \cdot Q_2(x)$$

We calculate $(x - x_1) \cdot (x - x_2)$ and eliminate i, we denote the obtained form for convenience of writing with R(x). Therefore $P_4(x) = R(x) \cdot Q_2(x)$. We find $Q_2(x)$ by dividing $P_4(x)$ by $Q_2(x)$. We solve $Q_2(x) = 0$ and find from here x_3, x_4. We know x_1 from the statement as being x=a+bi, and x_2 we know from theory as being x=a-bi.

Example 2: Determine a and b, then solve the equation

$$x^4 - 7x^3 + 21x^2 + ax^1 + b = 0$$

knowing that it admits the root x=1+2i

Theory: If a polynomial P(X) admits the root x=a, then P(a)=0 (Bézout's Theorem).

Idea: Since P(x) admits the roots x=a and x=b, according to Bézout's theorem, we can write that P(a)=0 and P(b)=0. We obtained a system of two equations with two unknowns, which we solve and find a and b. Now we know the form of P(x) and we use the theory according to which if the polynomial admits the root x=1+2i, it means it also admits the root x=1-2i. We proceed as in the case of the previous problem and find the other roots.

Type 6. Equations that Admit Root $x = a + \sqrt{b}$

Example: Solve the equation $x^4 - 4x^3 + x^2 + 6x^1 + 2 = 0$ knowing that it admits the root $x = 1 - \sqrt{2}$.

Theory: If an equation admits the root $x = a + \sqrt{b}$, then it also admits the root $x = a - \sqrt{b}$

Idea: Similar to the previous problem. Let for example the polynomial $P_4(x)$ which we know admits the root $x_1 = a + \sqrt{b}$. According to the theory, $x_2 = a - \sqrt{b}$. This means that we can write

$$P_4(x) = (x - x_1) \cdot (x - x_2) \cdot Q_2(x)$$

We calculate $(x - x_1) \cdot (x - x_2)$ and eliminate $\sqrt{b}$, we denote the obtained form for convenience of writing with R(x). Therefore $P_4(x) = R(x) \cdot Q_2(x)$. We find $Q_2(x)$ by dividing $P_4(x)$ by $Q_2(x)$. We solve $Q_2(x) = 0$ and find from here x_3, x_4. We know x_1 from the statement as being $x = a + \sqrt{b}$, and x_2 we know from theory as being $x = a - \sqrt{b}$.

SHEET 18: Higher Degree Equations

Part II

Type 7. Equations with Maximum Degree Coefficient =1

Example: Solve the equation $x^4 - 2x^3 - 5x^2 + 8x + 4 = 0$

Theory: The integer roots of the equation **could be** among the divisors of the constant term.

Idea: Extract the divisors of the constant term, e.g.: +1,-1,+2,-2,+4,-4 and verify in turn if they are roots with Bézout's theorem. That is, verify if P(+1)=0. If it equals zero, it means it is a root, otherwise it is not. Make verifications for all divisors of the constant term. If we find for example two roots, let x_1 and x_2, we write $P(x) = (x - x_1)(x - x_2)Q_2(x)$. We calculate $(x - x_1)(x - x_2)$ and obtain a second degree equation, we denote it for convenience of writing with R(x). We divide P(x) by R(x) and obtain $Q_2(x)$. We solve $Q_2(x) = 0$ and find x_3 and x_4.

Type 8. General Case = Equations with Maximum Degree Coefficient Different from 1 (Also valid for maximum degree coefficient equal to 1 as a particular case)

Example: Solve the equation $6x^4 - 17x^3 - x^2 + 8x - 2 = 0$

Theory: The roots of the equation **could be** of the form $\alpha = \frac{p}{q}$, p=divisor of the constant term, and q=divisor of the maximum degree coefficient.

Idea: The method involves many calculations. Extract the divisors of the constant term, e.g.: p= +1,-1,+2,-2 and the divisors of the maximum rank coefficient e.g.: q=+1,-1,+2,-2,+3,-3,+6,-6. Form all combinations of type $\alpha = \frac{p}{q}$, namely $\frac{+1}{+1}, \frac{+1}{-1}, \frac{+1}{+2} \cdot \cdot$, etc. and verify with Bézout's theorem if $P(\alpha)$=0. If two solutions are found, proceed further as in the case of the previous problem.

Type 9. Binomial Equations

Example: Solve the equation $3x^7 = 5$

Theory: Bring the equation to the form $x^n = a$ and write the number a as a complex number, of the form $r(cos(\alpha) + i \cdot sin(\alpha))$. Use if necessary the forms $1 = cos(0) + i \cdot sin(0)$, respectively $-1 = cos(\pi) + i \cdot sin(\pi)$. We obtain $x^n = r(cos(\alpha) + i \cdot sin(\alpha))$ and apply the formula for the radical of a complex number and obtain the roots:

$$x_k = \sqrt[n]{r}(cos\frac{\alpha + 2k\pi}{n} + i \cdot sin\frac{\alpha + 2k\pi}{n}), k = 0, 1 \cdots k - 1$$

For example for $3x^7 = 5$ we write $x^7 = \frac{5}{3} \cdot 1$, that is $x^7 = \frac{5}{3} \cdot (cos(0) + i \cdot sin(0))$ therefore

$$x_k = \sqrt[n]{\frac{5}{3}}(cos\frac{0 + 2k\pi}{5} + i \cdot sin\frac{0 + 2k\pi}{5}), k = 0, 1 \cdots 4$$

Type 10. Other Methods for Solving Higher Degree Equations

- Using Bézout's theorem, for multiple roots (10th grade material)
- Using Viète's relations (10th grade material)
- Using Rolle's sequence (11th grade material, involves derivatives)
- Graphical solution (11th grade material, sometimes involves derivatives)

EPILOGUE

You have completed Worksheets 1 through 18 — the essential foundations of rigorous mathematics. If you worked through this material seriously, you now possess something valuable: a solid base upon which everything else can be built.

What You Have Accomplished

These eighteen worksheets represent the **core mathematical toolkit** that every serious student needs. You have covered:

- Geometry and trigonometry — the language of shapes and angles
- Complex numbers — extending our number system
- Algebraic techniques — manipulating expressions with confidence
- Functions — understanding relationships and transformations
- Exponentials and logarithms — modeling growth and decay
- Combinatorics — counting with precision
- Polynomials and equations — solving systematically

This is not "basic" mathematics. This is **foundational** mathematics — the kind that separates those who merely pass exams from those who truly understand.

If you have mastered these concepts, you are ready for whatever comes next: more advanced mathematics, university studies, technical applications, or simply the satisfaction of knowing you built something solid.

The Compact Philosophy

This book deliberately focused on **essentials only**. Many topics were excluded — not because they are unimportant, but because the goal was to give you a **complete, usable foundation** without overwhelming you.

After 45 years of teaching, one lesson is clear: **mastery of fundamentals beats superficial exposure to everything**. You are better served by deeply understanding eighteen core topics than by barely remembering thirty-four.

This compact edition represents that philosophy. You now have a **solid core**. Everything else can be added later, if needed, on top of this foundation.

What Comes Next?

For many students, these eighteen worksheets are **sufficient**. They cover what most international curricula require for high school mathematics. If your goal was to build a solid foundation, review for exams, or help your children with their studies — you have achieved that goal.

For students seeking **advanced preparation** — university entrance exams, technical studies, or deeper mathematical exploration — an **Advanced Edition** is available. It includes these same eighteen worksheets plus an additional sixteen covering:

- Linear algebra (determinants, matrices, systems of equations)
- Limits and continuity (sequences and functions)
- Derivatives and their applications
- Integrals and their applications
- Introduction to algebraic structures

The Advanced Edition represents the complete curriculum of rigorous European high schools (grades 9–12) and prepares students for technical university programs.

You do not need it unless your path requires these advanced topics. This Concise Edition is **complete in itself** for standard mathematics education.

For information about the Advanced Edition, visit: **rjurcone.net** or contact: **jurcone@yahoo.com**

A Final Word: On Persistence

Mathematics is not about being naturally gifted. It is about **showing up consistently** and doing the work, even when it feels difficult.

If you completed these eighteen worksheets, you proved something important: **you can learn hard things when you persist**.

That lesson extends far beyond mathematics. The discipline you developed working through these problems — the patience to struggle with confusion, the persistence to try again after mistakes, the satisfaction of finally understanding — transfers to everything else you attempt in life.

You are stronger than you were when you started.

Whether you continue to advanced mathematics or apply these skills elsewhere, remember this: the book gave you the tool, but **you** did the hard work. You showed up. You persisted. You learned.

That achievement belongs to you alone.

A final reflection:

Years from now, you will have forgotten most of the formulas in this book. That is natural and acceptable. Formulas can always be looked up when needed.
But if this book succeeded in its deepest purpose, you will **not** have forgotten the courage you built — the willingness to face difficulty, to persist through confusion, to try again after failure.
That courage, though invisible in these pages, may be the most important thing this book has to offer. It will serve you far beyond mathematics, in every challenge life presents.

In truth, you have only one task, and if you accomplish it, everything else will follow:

Only be strong and courageous.

With respect,
The Author